MS8 Dispersive Hydrodynamics

Organisers: G.A. El and M.A. Hoefer

Titles and Abstracts

- **Tom Bridges** (Surrey University, UK)
  On coalescing characteristics in Whitham modulation theory: the characteristic (Krein) signs and their nonlinear implications

  Abstract: The change of type of the Whitham modulation equations from elliptic to hyperbolic is signaled, in the linearization, by a coalescence of purely imaginary eigenvalues, much like the Hamiltonian Hopf bifurcation. It looks like two eigenvalues with opposite Krein signature meeting and becoming unstable. However, there is no natural Hamiltonian structure. Instead we find that the linearization is more naturally interpreted as a Hermitian matrix pencil so the invariant is the "characteristic sign" which is more general than Krein signature in that zero eigenvalues also have a sign. This theory becomes more elaborate and interesting for multiphase Whitham modulation theory and is essential for studying coalescence of characteristics. The theory is formulated, the sign defined, and then the nonlinear implications are studied. By rescaling and reformulating near coalescing characteristcs it is found that the nonlinear behaviour is governing by a two-way Boussinesq equation. An example is the coalescence of characteristics in the family of two-phase wavetrains in the coupled nonlinear Schrodinger equation.

- **Wooyoung Choi** (New Jersey Institute of Technology, USA)
  On resonant interactions of gravity-capillary waves without energy exchange

  Abstract: We consider resonant triad interactions of gravity-capillary waves. Using the amplitude equations recovered from a Hamiltonian formulation for water waves, it is shown that any resonant triad can interact without energy exchange if the initial wave amplitudes and relative phase satisfy two conditions for fixed point solutions of the amplitude equations to exist. Furthermore, it is shown that the symmetric resonant triad exchanging no energy forms a transversely-modulated traveling wave field, which can be considered a two-dimensional generalization of Wilton ripples.

- **Thibault Congy** (Loughborough University and Northumbria University, UK)
  Nonlinear Schrödinger equation and the universal description of dispersive shock waves

  Abstract: The nonlinear NLS equation and the Whitham modulation equations both describe slowly varying, locally periodic nonlinear wavetrains, albeit in differing amplitude-frequency domains. In this paper, we take advantage of the overlap regime for the applicability of the NLS equation and the Whitham modulation theory to develop universal analytical description of dispersive shock waves (DSWs) generated in Riemann problems for a broad class of integrable and non-integrable nonlinear dispersive equations. The proposed method extends and complements the DSW fitting theory prescribing the motion of DSW edges. Our approach also provides a natural framework to analyse DSW stability. We consider several representative physically relevant examples illustrating efficacy of the developed general theory. Comparisons with direct numerical simulations show that inclusion of

higher order terms in the NLS equation enables a remarkably accurate description of the DSW modulation in a broad vicinity of the harmonic, small amplitude, edge.
This is joint work with Gennady El, Mark Hoefer and Michael Shearer

- **Alexander Dyachenko** (Landau Institute for Theoretical Physics, Russia)
  Canonical system of equations for 1D water wave and its NLSE approximation.

  Abstract: A system of two one-dimensional equations and the Hamiltonian for bidirectional deep-water waves are obtained. The system is obtained using canonical transformation that eliminates all non-resonant (three- and four-wave) terms in the original Hamiltonian. Simplest solutions are obtained, one of which describes a nonlinear standing wave. For the case of weakly modulated waves an approximate system of two NLS type equations is obtained, and its particular exact solutions are found.

- **Ted Johnson** (University College London, UK)
  Wavepackets as solutions of the Ostrovsky equation

  Abstract: The Whitham modulation equations for the Ostrovsky equation are used to analyse localised cnoidal wavepacket solutions of the Ostrovsky equation in the weak rotation limit. The analysis is split into two main parameter regimes: the Ostrovsky equation with normal dispersion relevant to typical oceanic parameters and the Ostrovsky equation with anomalous dispersion relevant to strongly sheared oceanic flows and other physical systems. For anomalous dispersion a new steady, symmetric cnoidal wavepacket solution is presented. The new wavepacket can be represented as a solution of the modulation equations and an analytical solution for the outer solution of the wavepacket is given. For normal dispersion the modulation equations are used to describe the unsteady finite-amplitude wavepacket solutions produced from the rotation-induced decay of a Korteweg-de Vries solitary wave. Again, an analytical solution for the outer solution can be given. The centre of the wavepacket closely approximates a train of solitary waves with the results suggesting that the unsteady wavepacket is a localised, modulated cnoidal wavetrain. The formation of wavepackets from solitary wave initial conditions is considered, contrasting the rapid formation of the packets in anomalous dispersion with the slower formation of unsteady packets under normal dispersion.
  This is joint work with Ashley Whitfield

- **Mark Hoefer** (University of Colorado, Boulder, USA)
  Using Whitham theory to describe modulated solitary waves

  Abstract: Whitham modulation theory is a well-known asymptotic technique to describe the slow modulations of nonlinear, periodic, traveling wave solutions of nonlinear dispersive partial differential equations. The resultant first order, quasi-linear Whitham modulation equations can be obtained by averaging conservation laws for the governing partial differential equation over a period of the periodic wave family. This theory has been successfully utilized for the stability of periodic waves and the propagation of dispersive shock waves with many applications. This talk will focus upon the little-studied, singular, solitary wave (zero wavenumber) limit of the Whitham modulation equations. These equations describe a slowly varying mean flow that is decoupled from the solitary wave modulation. Interestingly, the solitary

wave modulation is described by an amplitude field that is spatially defined everywhere; solitary wave trajectories are characteristics. Applications include modulated solitary waves and the propagation of solitary waves through other nonlinear waves such as dispersive shock waves and rarefaction waves. Multiple governing partial differential equations and physical examples will be highlighted.

- **Noel Smyth** (University of Edinburgh, UK)
  Dispersive shock fitting for Benjamin-Ono type equations

  Abstract: While less well known than solitary waves, dispersive shock waves (DSWs), termed undular bores in fluid mechanics, are a nonlinear dispersive wave phenomenon with wide occurrence in nature, examples being tidal bores, tsunamis and morning glory clouds. A classic DSW is a modulated wavetrain which continually expands in time and which consists of solitary waves at one edge and linear dispersive waves at the other. Due to this unsteady, modulated nature, DSW solutions of nonlinear, dispersive wave equations are much more difficult to find than solitary wave solutions. The technique used to find DSW solutions is Whitham modulation theory, which is a general method for analysing slowly varying (modulated) wavetrains. If the modulation equations of this theory are hyperbolic, then the DSW solution is found as a centred simple wave solution. The problem is that the setting of the Whitham modulation equations in Riemann invariant form, so that a simple wave solution can be found, is only guaranteed if the original equation is integrable. To overcome this limitation to finding DSW solutions, the dispersive shock wave fitting method was developed. With this method the leading and trailing edges of a DSW, but not its interior, can be determined without knowledge of the full Whitham modulation equations. This method was originally developed for nonlinear dispersive wave equations with periodic wave solutions determined by odes of elliptic function form and DSWs of Korteweg-de Vries form. This seminar will outline the extension of the dispersive shock wave fitting method to equations with nonlocal dispersion of Benjamin-Ono type.
  This joint work with Gennady El and Khiem Nguen.

- **Antonio Moro** (Northumbria University, UK)
  Hydrodynamic theory of phase transitions

  Abstract: A phase transition denotes a drastic change of state of a thermodynamic system due to a continuous change of parameters. Inspired by the theory of nonlinear conservation laws and shock waves we develop an approach to phase transitions based on the solution of Maxwell relations. This theory provides an exact mathematical description of discontinuities of order parameters and phase transitions via nonlinear integrable PDEs, it allows to classify universal classes of equations of state and interpret the occurrence of critical behaviours in terms of the dynamics of nonlinear shock wave fronts.

  The approach is shown at work in the case of mean field magnetic and fluid models, nematic liquid crystals and random graphs

- **Stephane Randoux** (University of Lille, France)
  Optical random Riemann waves and the pre-breaking stage of integrable turbulence

  Abstract: We examine integrable turbulence in the framework of the defocusing cubic one-dimensional nonlinear Schrödinger equation. This is done theoretically and experimentally, by realizing an optical fiber experiment in which the defocusing Kerr nonlinearity strongly dominates linear dispersive effects. Using a dispersive-hydrodynamic approach, we show that the development of integrable turbulence can be divided into two distinct stages, the initial, pre-breaking stage being described by a system of interacting random Riemann waves. We explain the low-tailed statistics of the wave intensity in integrable turbulence and show that the Riemann invariants of the asymptotic nonlinear geometric optics system represent the observable quantities that provide new insight into statistical features of the initial stage of the integrable turbulence development by exhibiting stationary probability density functions.
  This is joint work with Francois Gustave, Gennady El and Pierre Suret

- **Patrick Sprenger** (University of Colorado, Boulder, USA)
  Soliton-mean flow interactions in bi-directional dispersive hydrodynamic systems

  Abstract: Solitary waves, sometimes referred to as solitons, are localized traveling pulses that are ubiquitous in dispersive fluid-like media. We consider the interaction of a solitary wave with a large scale, spatially extended hydrodynamic flow that results from a hydraulic transition in the far-field boundary conditions. As a specific example, we consider a defocusing optical medium described by the nonlinear Schrödinger equation. The hydraulic transition generically gives rise to a rarefaction waves (RWs) or dispersive shock wave (DSWs). Under the scale separation assumption of nonlinear wave (Whitham) modulation theory, the nontrivial nonlinear interaction between the soliton and the evolving hydrodynamic wave is described in terms of simple wave solutions to an asymptotic reduction of the Whitham equations. The simple wave solution of the reduced system ultimately quantifies the effect of the hydrodynamic wave on the soliton and gives explicit expressions for the soliton's amplitude and trajectory and gives conditions for the soliton becoming trapped inside or transmitted through the hydrodynamic flow.

Development of undular bores in a variable medium

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In the weakly nonlinear long wave regime many physical systems can be modelled using the variable-coefficient Korteweg-de Vries equation,

$$A_t + cA_x + \frac{cQ_x}{2Q}A + \mu AA_x + \lambda A_{xxx} = 0\,. \tag{1}$$

Here $A(x,t)$ is the amplitude of the linear long wave mode with linear phase speed c, Q is a linear magnification factor needed to ensure that wave action flux is conserved, and μ, λ are system-dependent coefficients. In the most general case c, Q, μ, λ vary slowly in space and time. Here we describe application of this model to the the transformation of an internal internal bore propagating over variable topography. Particular attention is paid to the passage through a critical point where there is a polarity change. The analysis is based on an adaptation of the Whitham modulation theory to a solitary wave train.

Title: Riemann Problems for the BBM Equation

Speaker: Michael Shearer, North Carolina State University

Abstract

The BBM equation

$$u_t + (\tfrac{1}{2}u^2)_x = u_{xxt} \tag{1}$$

is a variation on the KdV equation that has interestingly different properties from the KdV equation due to the rate dependent dispersive term on the right side. In this talk, I summarize what we know from analysis and numerical simulations concerning solutions of Riemann problems, meaning initial value problems for equation (1) with piecewise constant jump initial data:

$$u(x,0) = \begin{cases} u_L & x < 0, \\ u_R & x > 0. \end{cases} \tag{2}$$

Numerical results show complicated wave combinations for some initial data u_L, u_R. Analysis using refinements of the Whitham modulaton theory help explain thresholds between different types of behavior. Asymptotics allow reduction to the system of shallow water equations and further analysis relates the structure of waves to a degenerate version of the Nonlinear Schrödinger equation. This is joint work with Gennady A. El and Mark A. Hoefer.